Imperial College
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Topic 4

Kirchoff's Laws & Nodal Analysis

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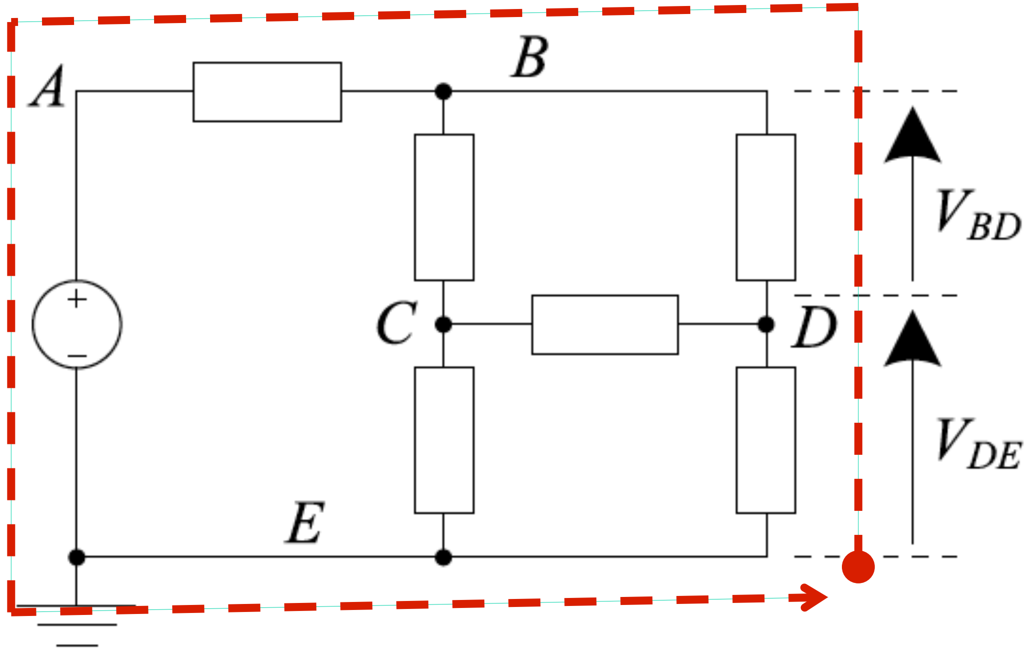
Kirchoff's Voltage Law

- The five nodes are labelled A, B, C, D, E where E is the reference node.
- Each component that links a pair of nodes is called a *branch* of the network.



Kirchoff's Voltage Law (KVL)

The sum of the voltage changes around any closed loop is zero.

- ***KVL*** is a consequence of the fact that the work done in moving a charge from one node to another does not depend on the route you take; in particular the work done in going from one node back to the same node by any route is zero.
- Example: $V_{DE} + V_{BD} + V_{AB} + V_{EA} = 0$
- Equivalent formulation:

$$V_{XY} = V_{XE} - V_{YE} = V_X - V_Y$$ for any nodes X and Y .

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Kirchoff's Current Law

- Wherever charges are free to move around, they will move to ensure charge neutrality everywhere at all times.
- A consequence is ***Kirchoff's Current Law (KCL)*** which says that the current going into any closed region of a circuit must equal the current coming out.

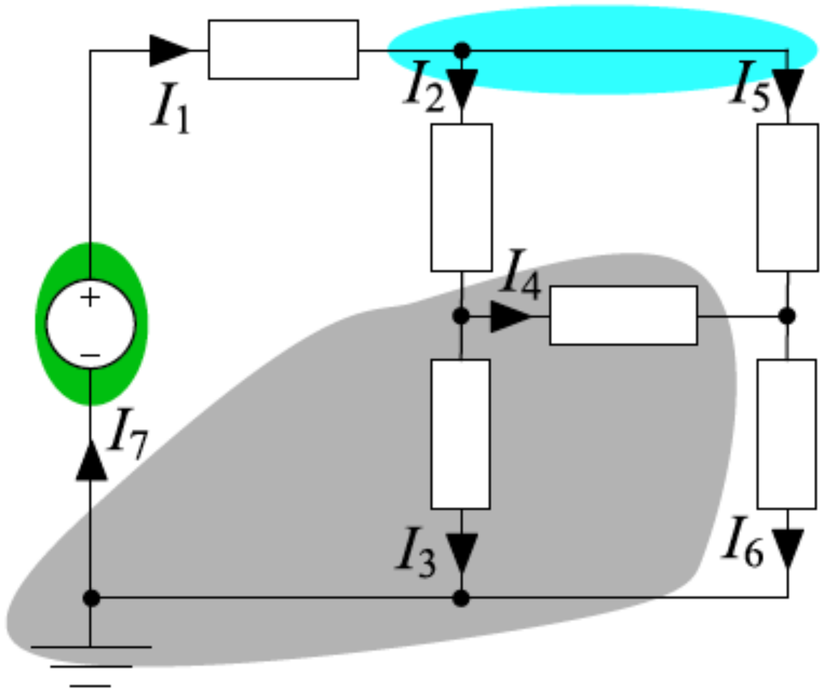
Kirchoff's Current Law (KCL)

The currents flowing out of any closed region of a circuit sum to zero.

Green: $I_1 = I_7$

Blue: $-I_1 + I_2 + I_5 = 0$

Gray: $-I_2 + I_4 - I_6 + I_7 = 0$

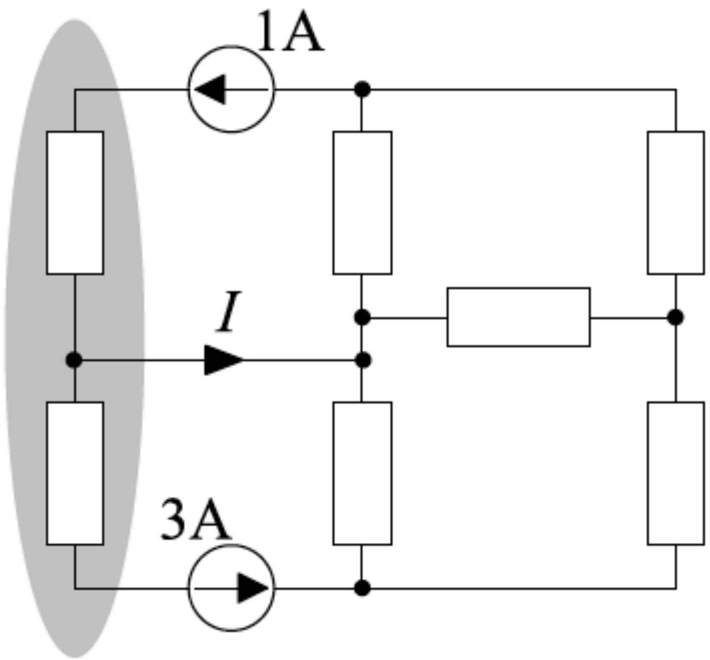


KCL Example

- The currents and voltages in any **linear circuit** can be determined by using KCL, KVL and Ohm's law.
- Sometimes KCL allows you to determine currents very easily without having to solve any simultaneous equations:
- How do we calculate I ?

KCL: $-1 + I + 3 = 0$

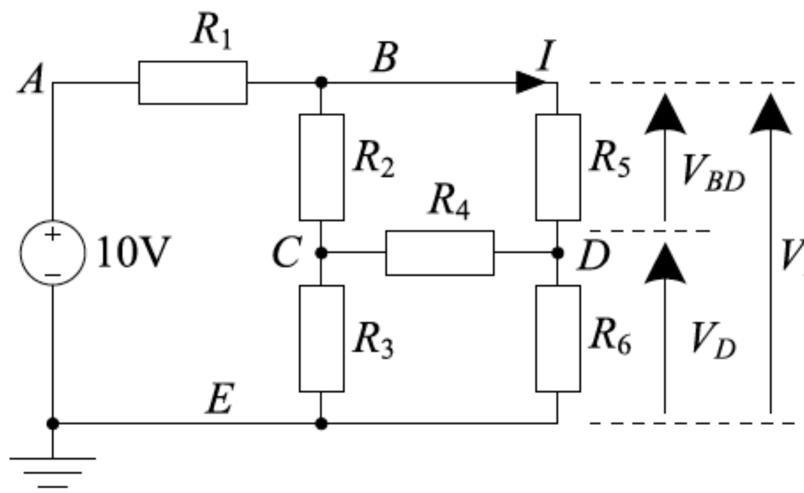
$\Rightarrow I = -2\text{A}$



- Note that here I ends up negative which means we chose the wrong arrow direction to label the circuit. This does not matter. You can choose the directions arbitrarily and let the algebra take care of reality.

Aim of Nodal Analysis

- The aim of nodal analysis is to determine the voltage at each node relative to the **reference node** (or ground). Once you have done this you can easily work out anything else you need.
- There are two ways to do this:
  (1) Nodal Analysis - systematic; always works
  (2) Circuit Manipulation - ad hoc; but can be less work and clearer



Reminders:

- A node is all the points in a circuit that are directly interconnected.
- We assume the interconnections have zero resistance so all points within a node have the **same voltage**. Five nodes: A, ... ,E.

Ohm's Law: $V_{BD} = I R_5$

KVL: $V_{BD} = V_B - V_D$

KCL: Total current exiting (or entering) any closed region is zero.

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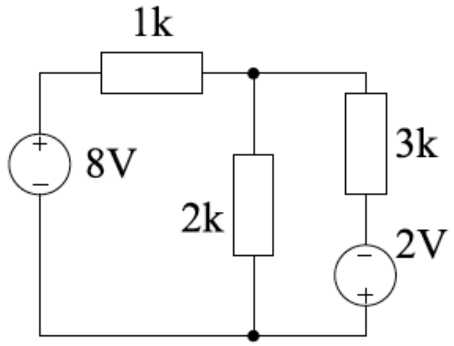
Nodal Analysis Stage 1: Label Nodes

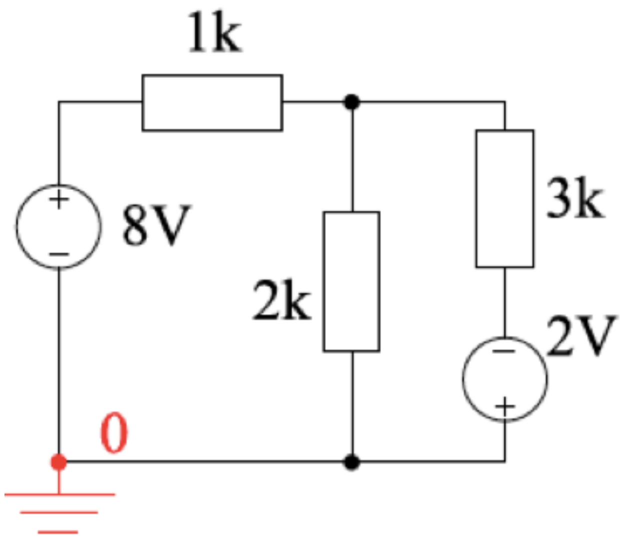
- To find the voltage at each node, the first step is to label each node with its voltage as follows:

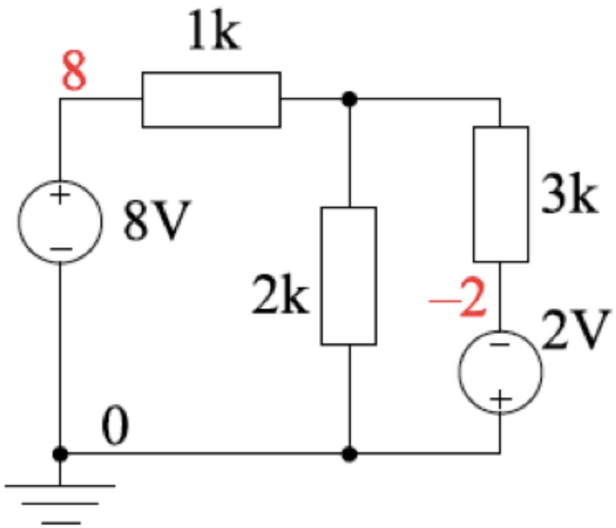


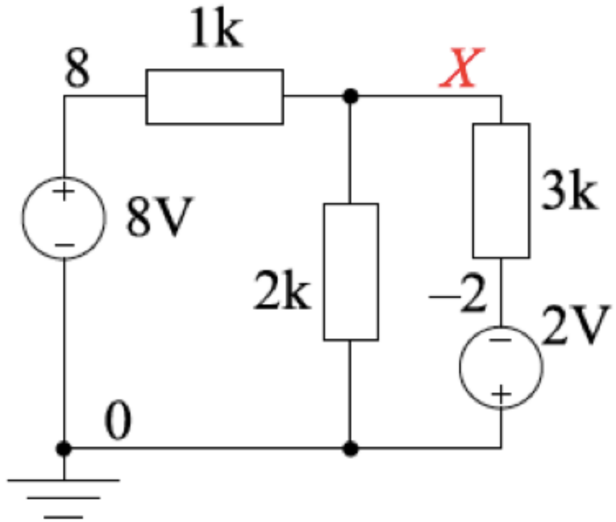
(1) Pick any node as the voltage reference. Label its voltage as $0V$.

(2) If any fixed voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.

(3) Pick an unlabelled node and label it with $X, Y, \ldots$, then go back to step (2) until all nodes are labelled.

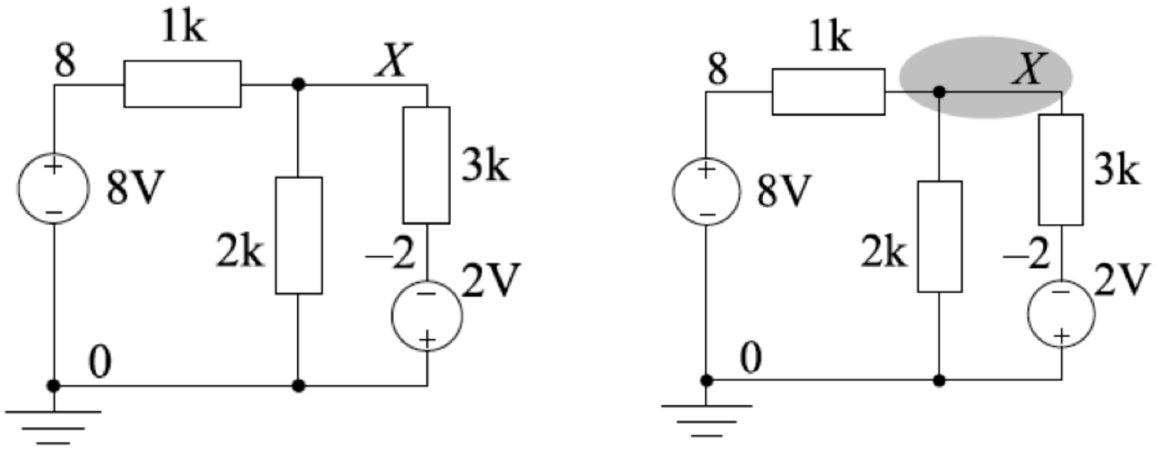






Nodal Analysis Stage 2: KCL Equations

- The second step is to write down a KCL equation for each node labelled with a variable by setting the total current flowing out of the node to zero.
- For a circuit with N nodes and S voltage sources you will have $N - S - 1$ simultaneous equations to solve.



- We only have one variable:

$$\frac{X-8}{1\text{ k}} + \frac{X-0}{2\text{ k}} + \frac{X-(-2)}{3\text{ k}} = 0 \quad \Rightarrow \quad (6X - 48) + 3X + (2X + 4) = 0$$

$$11X = 44 \quad \Rightarrow \quad X = 4$$

Numerator for a resistor is always of the form $X - V_N$ where V_N is the voltage on the other side of the resistor.

Current Sources

- Current sources cause no problems.

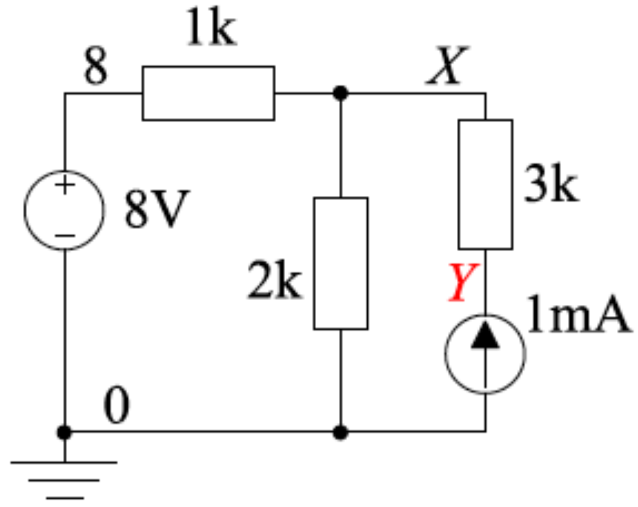
  (1) Pick reference node.

  (2) Label nodes: 8, X and Y .

  (3) Write equations

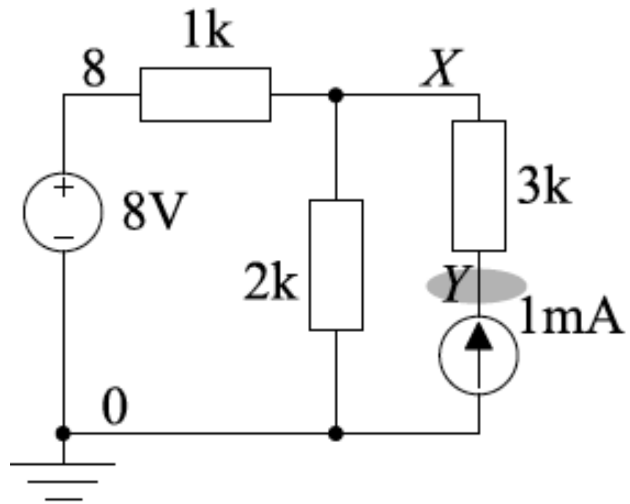
$$\frac{X-8}{1}+\frac{X}{2}+\frac{X-Y}{3}=0$$

$$\frac{Y-X}{3}+(-1)=0$$



Ohm's law works OK if all resistors are in kΩ and all currents in mA.

(4) Solve the equations: X = 6, Y = 9



Floating Voltage Sources

- ***Floating voltage sources*** have neither end connected to a known fixed voltage. We have to change how we form the KCL equations slightly.

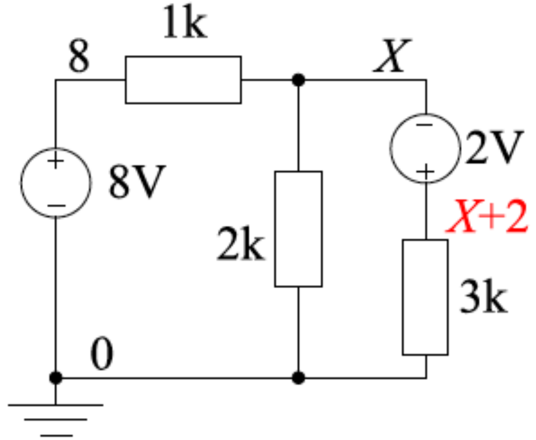
  (1) Pick reference node.

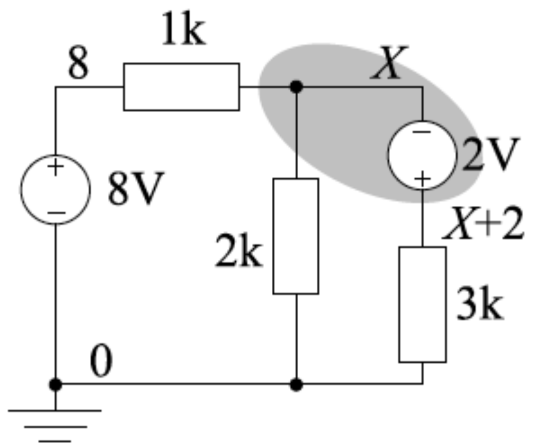
  (2) Label nodes: 8, X and $X + 2$ since it is joined to X via a voltage source.

  (3) Write KCL equations but count all the nodes connected via floating voltage sources as a single "*super-node*" giving one equation

$$\frac{X-8}{1} + \frac{X}{2} + \frac{(X+2)-0}{3} = 0$$

  (4) Solve the equations: X = 4



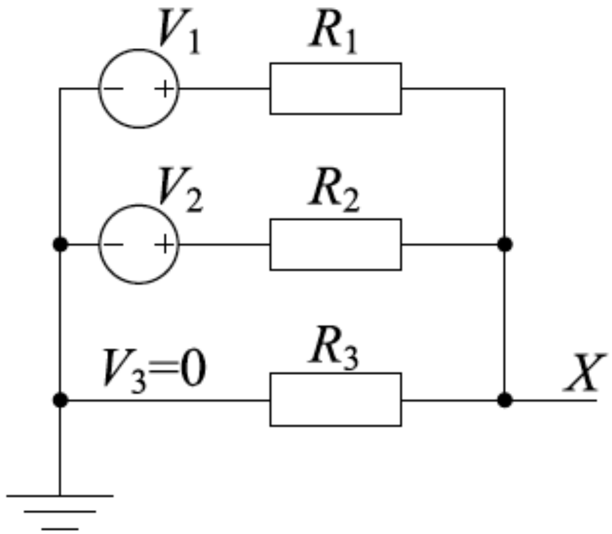


Ohm's law always involves the difference between the voltages at either end of a resistor. (Obvious but easily forgotten)

Weighted Average Circuit

- A very useful sub-circuit that calculates the weighted average of any number of voltages.

- KCL equation for node X:



$$\frac{X-V_1}{R_1}+\frac{X-V_2}{R_2}+\frac{X-V_3}{R_3}=0$$

- Still works if $V_3 = 0$.

- Or using conductances:

$$(X-V_1)G_1+(X-V_2)G_2+(X-V_3)G_3=0$$

$$X(G_1+G_2+G_3)=V_1G_1+V_2G_2+V_3G_3$$

$$X=\frac{V_1G_1+V_2G_2+V_3G_3}{G_1+G_2+G_3}=\frac{\sum V_iG_i}{\sum G_i}$$

Voltage X is the average of V1, V2, V3 weighted by the conductances.

Simple Digital to Analogue Converter

A 3-bit binary number, b, has bit-weights of 4, 2 and 1. Thus 110 has a value 6 in decimal. If we label the bits $b_2b_1b_0$, then $b = 4b_2 + 2b_1 + b_0$.

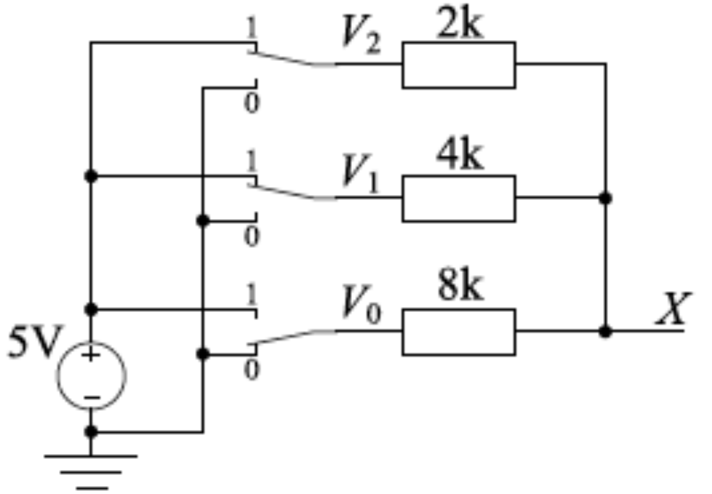
We use $b_2b_1b_0$ to control the switches which determine whether $V_i = 5\,\mathrm{V}$ or $V_i = 0\,\mathrm{V}$. Thus $V_i = 5b_i$. Switches shown for $b = 6$.

$$X = \frac{\frac{1}{2}V_2 + \frac{1}{4}V_1 + \frac{1}{8}V_0}{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}$$

$$= \tfrac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$

but $V_i = 5 \times b_i$ since it connects to either $0\,\mathrm{V}$ or $5\,\mathrm{V}$

$$= \tfrac{5}{7}\left(4b_2 + 2b_1 + b_0\right) = \tfrac{5}{7}b$$

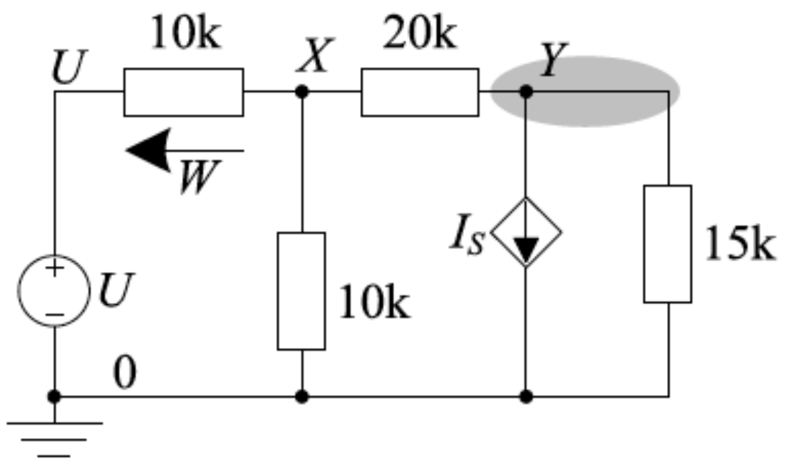


$$G_2 = \frac{1}{R_2} = \frac{1}{2\,k} = \frac{1}{2}\,\mathrm{mS},\ \ldots$$

So we have made a circuit in which X is proportional to a binary number b.

Dependent Voltage Sources

- A *dependent* voltage or current source is one whose value is determined by voltages or currents elsewhere in the circuit. These are most commonly used when modelling the behaviour of transistors or op-amps. Each dependent source has a *defining equation*.
- In this circuit: $I_S = 0.2W$ mA where W is in volts.



(1) Pick reference node.

(2) Label nodes: $0, U, X$ and Y .

(3) Write equation for the dependent source, I_S, in terms of node voltages:

$$I_S = 0.2\,(U - X)$$

(4) Write KCL equations:

$$\frac{X-U}{10} + \frac{X}{10} + \frac{X-Y}{20} = 0 \qquad \frac{Y-X}{20} + I_S + \frac{Y}{15} = 0$$

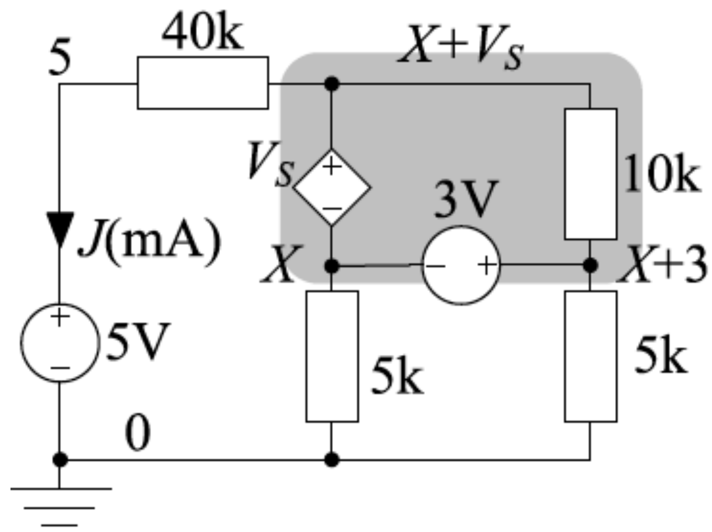
(5) Solve all three equations to find X, Y and IS in terms of U:

$$X = 0.1U,\ Y = -1.5U,\ I_S = 0.18U$$

Note that the value of U is assumed to be known.

Dependent Voltage Sources

- The value of the highlighted dependent voltage source is $V_S = 10J$ Volts where J is the indicated current in mA.



(1) Pick reference node.

(2) Label nodes: $0, 5, X, X+3$ and $X + V_S$.

(3) Write equation for the dependent source, V_S, in terms of node voltages:

$$V_S = 10J = 10 \times \frac{X+V_S-5}{40} \Rightarrow 3V_S = X - 5$$

(4) Write KCL equations: all nodes connected by floating voltage sources and all components connecting these nodes are in the same "super-node"

$$\frac{X+V_S-5}{40} + \frac{X}{5} + \frac{X+3}{5} = 0$$

(5) Solve the two equations: $X = -1$ and $V_S = -2$

Universal Nodal Analysis Algorithm

① Pick any node as the voltage reference. Label its voltage as 0 V. Label any dependent sources with VS, IS, ...

② If any voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.

③ Pick an unlabelled node and label it with X, Y, . . ., then loop back to step (2) until all nodes are labelled.

④ For each dependent source, write down an equation that expresses its value in terms of other node voltages.

⑤ Write down a KCL equation for each “normal” node (i.e. one that is not connected to a floating voltage source).

⑥ Write down a KCL equation for each “super-node”. A super-node consists of a set of nodes that are joined by floating voltage sources and includes any other components joining these nodes.

⑦ Solve the set of simultaneous equations that you have written down.

Summary

- Nodal Analysis
  - Simple Circuits (no floating or dependent voltage sources)
  - Floating Voltage Sources
    - use supernodes: all the nodes connected by floating voltage sources (independent or dependent)
  - Dependent Voltage and Current Sources
    - Label each source with a variable
    - Write extra equations expressing the source values in terms of node voltages
    - Write down the KCL equations as before